

q -EXTENSION OF THE p -ADIC GAMMA FUNCTION. II

BY

NEAL KOBLITZ¹

ABSTRACT. Taylor series and asymptotic expansions are developed for q -extensions of the p -adic psi (derivative of log-gamma) function “twisted” by roots of unity. Connections with p -adic L -functions and q -expansions of Eisenstein series are discussed. The p -adic series are compared with the analogous classical expansions.

Introduction. We shall study q -extensions of the ψ -function (derivative of log-gamma) and its “twists” (by roots of unity, Dirichlet characters, etc.) in the complex analytic, and especially the p -adic analytic cases. Using expressions for these functions as convolution transforms, we derive two types of expansions for them: Taylor expansions near $x = 0$ (or $x = 1$), and Stirling series for x large. For the usual type of ψ -function (which is the limit of the q -extension as $q \rightarrow 1$ in both the classical and p -adic cases), the coefficients in the Taylor and Stirling series are essentially the values of the Riemann zeta-function (or Dirichlet L -functions) at positive integers (Taylor series) and at negative integers (Stirling series). For the q -extensions, these coefficients involve Eisenstein series, as well as values of zeta- or L -functions; in this context, the k th normalized Eisenstein series G_k , for variable k , play the role of q -extensions of $\zeta(k)$.

In the complex analytic case this occurrence of G_k as Taylor coefficient, generalizing $\zeta(k)$, is related in a simple way to the appearance of $\frac{1}{2}\zeta(1-k)$ as the constant term in $G_k(q)$. Namely, a weight- k modular form $f(z)$, $q = e^{2\pi iz}$, satisfies $f(-1/z) = z^k f(z)$, so its behavior as $q = e^{2\pi iz} \rightarrow 1^-$ (i.e., as $z \rightarrow i0^+$) is directly determined by its behavior as $q \rightarrow 0^+$ (i.e., as $z \rightarrow i\infty$). In the p -adic case, the connection between the Taylor coefficient, which is a function of q near 1, and the q -expansion of the corresponding p -adic modular form, which is a function of q near 0, is more indirect. The connection is by p -adic analytic continuation, a purely analytic procedure which does not, so far as we know, admit any interpretation in terms of moduli of elliptic curves.

The p -adic construction of the convolution transform requires us to twist the p -adic ψ_q by a number z outside the unit disc around 1, e.g., $z \in \sqrt[d]{\mathbb{F}_p}$, $z \neq 1$, $p \nmid d$. (The use of the letter z in this context should not be confused with its use in $q = e^{2\pi iz}$.) Such a twist, sometimes called “regularization”, is routinely needed in order to make a bounded p -adic measure.

Received by the editors June 12, 1981.

1980 *Mathematics Subject Classification*. Primary 12B40; Secondary 33A15, 12B30, 10D30.

Key words and phrases. Gamma function, psi function, p -adic functions, q -extension, Stirling series, Eisenstein series.

¹Partially supported by NSF grant #MCS80-02271.

©1982 American Mathematical Society
0002-9947/81/0000-0625/\$06.00

In the complex analytic construction, one is not required to twist, and in fact, classical discussions of ψ and q - ψ functions generally treat the untwisted functions. However, the following classical examples illustrate that the twisted log-gamma and q -log-gamma functions have arisen naturally in the history of the subject.

EXAMPLE 1. Let $p > 2$ be a prime, let $S_n = \{j = 1, 2, \dots, pn - 1 \mid (\frac{j}{p}) = 1\}$ be the set of numbers less than pn which are squares modulo p (here $(\frac{\cdot}{p})$ is the Legendre symbol), and let $NS_n = \{j = 1, 2, \dots, pn - 1 \mid (\frac{j}{p}) = -1\}$ be the set of nonsquares.

Problem. Does the ratio

$$(0.1) \quad \prod_{j \in NS_n} j / \prod_{j \in S_n} j$$

approach a finite nonzero limit as $n \rightarrow \infty$, and, if so, what is it?

Solution. Using the formula

$$(0.2) \quad \Gamma(x) = \lim_{n \rightarrow \infty} \frac{(n-1)!n^x}{x(x+1) \cdots (x+n-1)},$$

we see that n to the power $\sum_{j=1}^{p-1} (\frac{j}{p}) \frac{j}{p}$ times (0.1) approaches the limit

$$(0.3) \quad \prod_{j \in S_1} \Gamma(j/p) / \prod_{j \in NS_1} \Gamma(j/p)$$

as $n \rightarrow \infty$. The exponent $\sum (\frac{j}{p}) \frac{j}{p}$ is zero if and only if $p \equiv 1 \pmod{4}$. If $p \equiv -1 \pmod{4}$ and $p > 3$, then this exponent is a negative integer equal to minus the class number of $Q(\sqrt{-p})$, according to a well-known formula of Dirichlet. Thus, in the former case (0.1) approaches the limit (0.3), while in the latter case it diverges to $+\infty$.

Here the logarithm of (0.3) is the value at 0 of the log-gamma function twisted by the quadratic Dirichlet character $\chi = (\frac{\cdot}{p})$,

$$(\log \Gamma)_\chi(x) = \sum_{\text{def}}^{p-1} \chi(j) \log \Gamma((x+j)/p).$$

The derivative of this function is typical of the twisted ψ -functions whose q -extensions we shall be studying.

EXAMPLE 2. More generally, for any nontrivial Dirichlet character χ modulo d , the value $(\log \Gamma)_\chi(0)$ is essentially the derivative at 0 of the corresponding Dirichlet L -series (see [23, p. 271]):

$$L'(0, \chi) = B_{1, \chi} \log d + (\log \Gamma)_\chi(0),$$

where $B_{1, \chi} = \frac{1}{d} \sum_{j=0}^{d-1} j \chi(j)$. ($B_{1, \chi} = 0$ if and only if $\chi(-1) = 1$.)

EXAMPLE 3. Among the simplest of the Rogers-Ramanujan identities are the following power series identities:

$$(0.4) \quad \begin{aligned} \prod_{n=0}^{\infty} (1 - q^{5n+1})^{-1} (1 - q^{5n+4})^{-1} &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}; \\ \prod_{n=0}^{\infty} (1 - q^{5n+2})^{-1} (1 - q^{5n+3})^{-1} &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}, \end{aligned}$$

where

$$(q)_n = (q; q)_n \stackrel{\text{def}}{=} (1-q)(1-q^2) \cdots (1-q^n)$$

(more generally,

$$(x; q)_n \stackrel{\text{def}}{=} (1-x)(1-xq) \cdots (1-xq^{n-1})).$$

Both sides of (0.4) converge if q is a real or complex parameter with $|q| < 1$. It can also be shown that the ratio of the first product to the second in (0.4) is equal to the continued fraction

$$1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}.$$

The identities (0.4) are equivalent to a statement about partitions (see [1, Chapter 7] for a detailed discussion).

The logarithm of the ratio of the two products on the left in (0.4) is a twisted version of the q -gamma function

$$(0.5) \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x} = \frac{(1-q)(1-q^2) \cdots}{(1-q^x)(1-q^{x+1}) \cdots} (1-q)^{1-x}.$$

Namely,

$$\log \frac{\prod_{n=0}^{\infty} (1-q^{5n+1})^{-1} (1-q^{5n+4})^{-1}}{\prod_{n=0}^{\infty} (1-q^{5n+2})^{-1} (1-q^{5n+3})^{-1}} = \sum_{j=1}^4 \left(\frac{j}{5} \right) \log \Gamma_{q^5} \left(\frac{j}{5} \right).$$

The right-hand side is the value at 0 of the q -log-gamma function twisted by the character $(\frac{j}{5})$, where one defines

$$(0.6) \quad (\log \Gamma_q)_\chi(x) \stackrel{\text{def}}{=} \sum_{j=0}^{d-1} \chi(j) \log \Gamma_{q^d}((x+j)/d),$$

for a character χ modulo d . It is the derivatives of functions like (0.6) which we shall be studying.

Although the classical and p -adic cases (the first and second sections of the present paper) are logically independent, there is a striking similarity between them. In fact, they are formally the same in the following sense. If the root of unity z used in the twist is replaced by z in the open unit disc about zero, then the same power series (in $\mathbb{Z}[[z, q]]$) occurs as Taylor coefficient classically and p -adically (except for “removal of the p -Euler factor” in the p -adic case). With z in this open disc the measures in the convolution transform are given by essentially the same formula in the classical and p -adic situations. Of course, the theory only has arithmetic or modular meaning when z is a root of unity; hence, the relation between the complex and p -adic functions is formal and indirect.

1. The classical case.

1. *Taylor series.* Let $\varphi(-x)$ be defined by convolution of a function $g(u)$ with a Stieltjes measure $df(u)$ on R^+ :

$$(1.1) \quad \varphi(x) = \int_0^\infty g(u+x) df(u).$$

We shall suppose in what follows that f and g are such that the integrals below converge.

Expanding

$$(1.2) \quad g(u+x) = \sum_{j=0}^{\infty} \frac{g^{(j)}(u)}{j!} x^j$$

(or, alternately, noting that $\varphi^{(j)}(0) = \int_0^\infty g^{(j)}(u) df(u)$), we have for x small:

$$(1.3) \quad \varphi(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \int_0^\infty g^{(j)}(u) df(u).$$

The ψ -function, twisted ψ -functions, q -extension of the ψ -function, and q -extensions of twisted ψ -functions are special cases of (1.1) with rather simple choices of f and g .

EXAMPLES. I. Let $f(u) = u - [u] - \frac{1}{2}$ be the first Bernoulli polynomial (made periodic with period 1), and let $g(u) = -1/u$. Start with the formula [23, p.261]

$$(1.4) \quad \log \frac{\Gamma(x)}{\sqrt{2\pi}} = \left(x - \frac{1}{2}\right) \log x - x - \int_0^\infty \frac{f(u) du}{u+x}.$$

Differentiating and replacing x by $x+1$, we obtain

$$\psi(1+x) = \log(1+x) - \frac{1}{2(1+x)} + \int_1^\infty \frac{f(u) du}{(u+x)^2}.$$

Integrating by parts, we have

$$(1.5) \quad \begin{aligned} \psi(1+x) &= \log(1+x) - \frac{1}{2(1+x)} + \frac{f(u)}{u+x} \Big|_{u=1^-} + \int_{1^-}^\infty \frac{df(u)}{u+x} \\ &= \log(1+x) + \int_{1^-}^\infty \frac{df(u)}{u+x}. \end{aligned}$$

To find the Taylor coefficients in the expansion of $\psi(1+x)$, we have $\psi(1) = -\gamma$, and for $j \geq 1$,

$$\begin{aligned} \frac{1}{j!} \psi^{(j)}(1) &= \frac{(-1)^{j-1}}{j} + \int_{1^-}^\infty (-1)^j \frac{df(u)}{u^{j+1}} \\ &= \frac{(-1)^{j-1}}{j} + (-1)^j \int_{1^-}^\infty \left(\frac{du}{u^{j+1}} - \frac{d[u]}{u^{j+1}} \right) \\ &= (-1)^{j-1} \int_{1^-}^\infty \frac{d[u]}{u^{j+1}}. \end{aligned}$$

But

$$\int_{1^-}^{\infty} \frac{d[u]}{u^{j+1}} = \sum_{n=1}^{\infty} \frac{1}{n^{j+1}} = \zeta(j+1).$$

This gives the expansion

$$(1.6) \quad \psi(1+x) = -\gamma + \sum_{j=2}^{\infty} (-1)^j \zeta(j) x^{j-1}.$$

Alternately, in the integral in (1.5) we could have expanded

$$g(u+x) = -1/(u+x) = \sum_{j=1}^{\infty} (-1)^j x^{j-1}/u^j$$

for $|x| < 1 \leq u$ as in (1.2).

Our reason for belaboring this derivation of a well-known expansion (see, e.g., [23, p. 241]) is that it is the prototype for the examples that follow and for our discussion of the p -adic case in §2.

II. Let $\rho: \mathbf{Z} \rightarrow \mathbf{C}$ be a periodic function of period d such that $\sum_{a=0}^{d-1} \rho(a) = 0$. The key examples are: (1) $\rho(j) = \chi(j)$, a nontrivial Dirichlet character modulo d ; and (2) $\rho(j) = z^j$, for $z \neq 1$ a d th root of unity. Let $g(u) = -1/u$ be the same as in Example I, and set $f_{\rho}(u) = \sum_{a=0}^{[u]} \rho(a)$.

We define the twisted ψ -function as follows:

$$(1.7) \quad \psi_{\rho}(x) \stackrel{\text{def}}{=} \frac{d}{dx} \sum_{a=0}^{d-1} \rho(a) \log \Gamma\left(\frac{x+a}{d}\right).$$

Using the formula (0.2) for $\Gamma(x)$, we easily see that

$$(1.8) \quad \psi_{\rho}(x) = - \int_{0^-}^{\infty} \frac{df_{\rho}(u)}{u+x}.$$

If, for example, $\rho = \chi$ is a nontrivial Dirichlet character, so that, in particular, $\rho(0) = 0$, then the integral really goes from 1^- to ∞ , and for $|x| < 1 \leq u$ we have

$$(1.9) \quad \psi_{\rho}(x) = \sum_{j=1}^{\infty} (-1)^j x^{j-1} \int_{1^-}^{\infty} \frac{df_{\chi}(u)}{u^j} = \sum_{j=1}^{\infty} (-1)^j x^{j-1} L(j, \chi),$$

since

$$\int_{1^-}^{\infty} u^{-j} df_{\chi}(u) = \sum_{n=1}^{\infty} \chi(n) n^{-j} = L(j, \chi).$$

In the example $\rho(j) = z^j$, we have $df_z(u-1) = z^{-1} df_z(u)$ for $u \geq 1$ (we use the notation $f_z = f_{\rho}$, $\psi_z = \psi_{\rho}$ here), so that by (1.8),

$$(1.10) \quad \begin{aligned} \psi_z(x+1) &= -\frac{1}{z} \int_{1^-}^{\infty} \frac{df_z(u)}{u+x} \\ &= \frac{1}{z} \sum_{j=1}^{\infty} (-1)^j x^{j-1} \int_{1^-}^{\infty} \frac{df_z(u)}{u^j} \\ &= \frac{1}{z} \sum_{j=1}^{\infty} (-1)^j x^{j-1} L(j, z), \end{aligned}$$

where

$$L(j, z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{z^n}{n^j}.$$

Note that (1.8), combined with the rule $df_z(u-1) = z^{-1}df_z(u)$, gives the functional relation

$$(1.11) \quad z\psi_z(x+1) - \psi_z(x) = 1/x,$$

which generalizes the relation $\psi(x+1) - \psi(x) = 1/x$ for the usual ψ -function. Combining (1.11) with the above Taylor series for $\psi_z(x+1)$ gives

$$(1.12) \quad \psi_z(x) = -\frac{1}{x} + \sum_{j=1}^{\infty} (-1)^j x^{j-1} L(j, z).$$

III. Let $0 < |q| < 1$, and take $f(u) = [u]$, $g(u) = (\log q)(q^u/(1 - q^u))$. The q -gamma function defined by (0.5) has the properties:

$$(1) \Gamma_q(x+1) = ((1 - q^x)/(1 - q))\Gamma_q(x),$$

$$(2) \Gamma_q(x) \rightarrow \Gamma(x) \text{ as } q \rightarrow 1^-.$$

The logarithmic derivative of $\Gamma_q(x)$ is, by (0.5),

$$(1.13) \quad \psi_q(x) = -\log(1 - q) + \log q \sum_{j=0}^{\infty} \frac{q^{x+j}}{1 - q^{x+j}}.$$

Hence,

$$(1.14) \quad \begin{aligned} \psi_q(1+x) &= -\log(1 - q) + \log q \sum_{j=1}^{\infty} \frac{q^{x+j}}{1 - q^{x+j}} \\ &= -\log(1 - q) + \int_{1^-}^{\infty} g(u+x) df(u). \end{aligned}$$

Note that for $q < 1$ the integral always converges, whereas in the limit when $q \rightarrow 1^-$ and $g(u)$ becomes $-1/u$ (Example I) one must modify $f(u)$ to make the integral converge.

Now define

$$(1.15) \quad P_k(q) = \left(q \frac{d}{dq} \right)^{k-1} \frac{q}{1 - q} = \sum_{n=1}^{\infty} n^{k-1} q^n,$$

so that $(1 - q)^k P_k(q) \in \mathbb{Z}[q]$. Since $g(u) = (\log q)(q^u/(1 - q^u))$, we have $g^{(k-1)}(u) = (\log q)^k P_k(q^u)$, so that for $k \geq 2$,

$$\frac{\psi_q^{(k-1)}(1)}{(k-1)!} = \frac{(\log q)^k}{(k-1)!} \int_{1^-}^{\infty} P_k(q^u) df(u);$$

this integral is equal to $\sum_{j=1}^{\infty} P_k(q^j) = \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m$, where

$$\sigma_{k-1}(m) \stackrel{\text{def}}{=} \sum_{d|m} d^{k-1}.$$

It is common to set $\psi_q(1) = -\gamma_q$ and call γ_q the q -Euler constant. Thus

$$\gamma_q = \log(1 - q) - \log q \sum_{j=1}^{\infty} \frac{q^j}{1 - q^j}.$$

Now for $k \geq 4$ an even integer, the weight- k Eisenstein series

$$(1.16) \quad E_k(z) = \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(mz + n)^k}$$

has the following expression in terms of $q = e^{2\pi iz}$ [22]:

$$(1.17) \quad E_k(z) = \frac{2(2\pi i)^k}{(k-1)!} G_k(q),$$

where

$$(1.18) \quad G_k(q) \stackrel{\text{def}}{=} \frac{1}{2} \zeta(1-k) + \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m = -\frac{B_k}{2k} + \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m,$$

in which B_k is the k th Bernoulli number:

$$(1.19) \quad \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

The definition (1.18) makes sense for any $k \neq 0$ (since $\zeta(1-k)$ and $\sigma_{k-1}(m) = \sum_{d|m} d^{k-1}$ make sense), but the relation to modular forms for $SL(2, \mathbb{Z})$ only holds for $k = 4, 6, 8, \dots$

Returning to ψ_q , we thus have

$$\psi_q(1+x) = -\gamma_q + \sum_{k=2}^{\infty} \frac{x^{k-1}}{(k-1)!} (\log q)^k \left(G_k(q) + \frac{B_k}{2k} \right).$$

Using (1.19) with t replaced by $x \log q$, we can rewrite this in the form

$$(1.20) \quad \psi_q(1+x) = -\gamma_q + \frac{1}{4} \log q - \frac{1}{2x} + \frac{1}{2} \frac{\log q}{q^x - 1} + \sum_{k=2}^{\infty} \frac{(\log q)^k}{(k-1)!} G_k(q) x^{k-1}.$$

Comparing with the Taylor expansion (1.6) for the usual ψ -function, we see that

$$(1.21) \quad \frac{(-\log q)^k}{(k-1)!} G_k(q) \quad \text{corresponds to } \zeta(k);$$

in particular,

$$(1.22) \quad \lim_{q \rightarrow 1^-} \frac{(-\log q)^k}{(k-1)!} G_k(q) = \zeta(k).$$

(Thus one might want to take $(-\log q)^s G_s(q) / \Gamma(s)$ as a q -extension of $\zeta(s)$.)

REMARKS. 1. For even $k \geq 4$, the relationship (1.22) follows from the fact that $G_k(0) = \frac{1}{2} \zeta(1-k)$ (see (1.18)). Namely, for $q = e^{2\pi iz}$, we have

$$\frac{(-\log q)^k}{(k-1)!} G_k(q) = \frac{(-2\pi iz)^k}{(k-1)!} G_k(q) = \frac{z^k}{2} E_k(z) = \frac{1}{2} E_k\left(-\frac{1}{z}\right)$$

by the modular property of E_k . But as $-1/z \rightarrow i\infty$ along the positive imaginary axis (i.e., as $q \rightarrow 1^-$), this gives

$$\frac{1}{2} E_k(i\infty) = \frac{1}{2} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n^k} = \zeta(k).$$

2. If one takes $(d^2/dx^2) \log$ of both sides of the relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, one obtains

$$(1.23) \quad \psi'(x) + \psi'(1-x) = \pi^2/\sin^2 \pi x.$$

The right side has the partial fraction decomposition

$$(1.24) \quad \frac{\pi^2}{\sin^2 \pi x} = (2\pi i)^2 \frac{e^{2\pi i x}}{(1 - e^{2\pi i x})^2} = \sum_{n \in \mathbf{Z}} \frac{1}{(x+n)^2},$$

while $\psi'(x)$ is equal to $\sum_{n=0}^{\infty} 1/(x+n)^2$. Thus, (1.23) expresses the fact that the sum for $\psi'(x)$ is “half” of the sum for $\pi^2 \csc^2 \pi x$.

The situation with the q -extension $\psi'_q(x)$ is similar. For simplicity, we take one more derivative, and we use (1.13) and (1.24) to write

$$(1.25) \quad \begin{aligned} \psi''_q(x) &= \frac{d}{dx} (\log q)^2 \sum_{j=0}^{\infty} \frac{q^{x+j}}{(1 - q^{x+j})^2} \\ &= (2\pi i z)^2 \sum_{j=0}^{\infty} \frac{d}{dx} \frac{e^{2\pi i z(x+j)}}{(1 - e^{2\pi i z(x+j)})^2} \\ &= z^2 \sum_{j=0}^{\infty} \frac{d}{dx} \sum_{n \in \mathbf{Z}} \frac{1}{(z(x+j) + n)^2} \\ &= -2 \sum_{j \geq 0, n \in \mathbf{Z}} \frac{1}{(x+j+n(-1/z))^3}; \end{aligned}$$

$$(1.26) \quad \psi''_q(1-x) = -2 \sum_{j < 0, n \in \mathbf{Z}} - \frac{1}{(x+j+n(-1/z))^3}.$$

So ψ''_q is “half” of the sum for $\wp'(x, -1/z)$ (where $\wp'$ denotes the derivative of the Weierstrass $\wp$ -function); that is, subtracting (1.26) from (1.25) gives

$$\frac{d^3}{dx^3} \log \Gamma_q(x) \Gamma_q(1-x) = -2 \sum_{j, n \in \mathbf{Z}} \frac{1}{(x+j+n(-1/z))^3} = \wp' \left(x, -\frac{1}{z} \right).$$

Thus, the expansion (1.20) for $\psi_q(x+1)$ is closely related to the expansion

$$(1.27) \quad \begin{aligned} \wp \left(x, -\frac{1}{z} \right) &= \frac{1}{x^2} + \sum_{k=4}^{\infty} E_k \left(-\frac{1}{z} \right) (k-1)x^{k-2} \\ &= \frac{1}{x^2} + \sum_{\substack{k \geq 4 \\ k \text{ even}}} \frac{2(\log q)^k}{(k-1)!} G_k(q) (k-1)x^{k-2}. \end{aligned}$$

Namely, if one adds (1.20) plus $(\log q) (q^x/(1-q^x))$ (which will give $\psi_q(x)$ on the left) to (1.20), with x replaced by $-x$, then the odd terms cancel in the sum on the right, and the derivative of the resulting expression is (up to a constant not containing x) equal to (1.27).

IV. Let ρ and f_ρ be as in Example II, and let $g(u) = (\log q)(q^u/(1-q^u))$ as in Example III. We define the q -extension of the twisted ψ -function (1.7) as follows:

$$(1.28) \quad \psi_{q,\rho}(x) \stackrel{\text{def}}{=} \frac{1}{d} \sum_{a=0}^{d-1} \rho(a) \psi_{q^d} \left(\frac{x+a}{d} \right).$$

By (1.13),

$$(1.29) \quad \psi_{q,\rho}(x) = \log q \sum_{j=0}^{\infty} \frac{\rho(j)q^{x+j}}{1-q^{x+j}} = \int_0^{\infty} g(u+x) df_{\rho}(u).$$

If $\rho = \chi$ is a nontrivial Dirichlet character, then for $|x| < 1 \leq u$,

$$\psi_{q,\chi}(x) = \sum_{k=1}^{\infty} x^{k-1} \frac{(\log q)^k}{(k-1)!} \int_1^{\infty} P_k(q^u) df_{\chi}(u),$$

where P_k is defined by (1.15). The integral here is equal to $\sum_{j=1}^{\infty} \rho(j)P_k(q^j) = \sum_{m=1}^{\infty} \sigma_{k-1,\chi}(m)q^m$, with

$$\sigma_{k-1,\chi}(m) \stackrel{\text{def}}{=} \sum_{d|m} \chi\left(\frac{m}{d}\right) d^{k-1}.$$

If $\rho(j) = z^j$, where $z \neq 1$ is a d th root of unity, then

$$\begin{aligned} \psi_{q,z}(x+1) &= \frac{1}{z} \int_1^{\infty} g(u+x) df_z(u) \\ &= \frac{1}{z} \sum_{k=1}^{\infty} x^{k-1} \frac{(\log q)^k}{(k-1)!} \int_1^{\infty} P_k(q^u) df_z(u), \end{aligned}$$

in which the integral is equal to

$$(1.30) \quad \sum_{m=1}^{\infty} \sigma_{k-1,z}(m)q^m, \quad \text{with } \sigma_{k-1,z}(m) \stackrel{\text{def}}{=} \sum_{d|m} z^{m/d} d^{k-1}.$$

In analogy to (1.11), $\psi_{q,z}$ satisfies the relationship $z\psi_{q,z}(x+1) - \psi_{q,z}(x) = -(\log q)(q^x/(1-q^x))$, which combines with the above Taylor series for $\psi_{q,z}(x+1)$ to give

$$(1.31) \quad \psi_{q,z}(x) = \log q \frac{q^x}{1-q^x} + \sum_{k=1}^{\infty} x^{k-1} \frac{(\log q)^k}{(k-1)!} \int_1^{\infty} P_k(q^u) df_z(u).$$

2. Stirling series. As before, let $\varphi(x)$ be of the form $\int_0^{\infty} g(u+x) df(u)$. Heuristically, we would like to consider x to be large, u to be small compared to x ; then expand $g(u+x)$ with x (rather than u) as the center: $g(u+x) = \sum (g^{(j)}(x)/j!)u^j$; and finally write

$$(1.32) \quad \varphi(x) \sim \sum_{j=0}^{\infty} \frac{g^{(j)}(x)}{j!} \int_0^{\infty} u^j df(u).$$

Unfortunately, for positive j , the integrals $\int_0^{\infty} u^j df(u)$ diverge in the examples I–IV above, although heuristically we might take, for example, $\int_0^{\infty} u^j d[u] = \sum n^j = \zeta(-j)$. Because of the divergence, we have to proceed somewhat differently to obtain the asymptotic series in the classical case; but we shall see that in the p -adic case there will be no convergence problem, and the p -adic version of (1.32) will be literally correct.

We now make the additional assumption that $f(u)$ is periodic of period d and

$$(1.33) \quad \int_0^d f(u) du = 0.$$

We write f as a Fourier series

$$f(u) = \sum_{0 \neq n \in \mathbf{Z}} a_n e^{2\pi i n u / d}.$$

Let $f^{(-1)}$ denote the integral of f normalized so that $\int_0^d f^{(-1)}(u) du = 0$, and define $f^{(-j)}$ inductively as $(f^{(-j+1)})^{(-1)}$, $j = 2, 3, \dots$. Then

$$f^{(-j)}(u) = \left(\frac{d}{2\pi i}\right)^j \sum_{0 \neq n \in \mathbf{Z}} \frac{a_n}{n^j} e^{2\pi i n u / d}.$$

In particular, $f^{(-j)}(0)$ is the value of the corresponding Dirichlet series:

$$(1.34) \quad f^{(-j)}(0) = \left(\frac{d}{2\pi i}\right)^j \sum_{0 \neq n \in \mathbf{Z}} \frac{a_n}{n^j}.$$

EXAMPLES REVISITED. I. If $f(u) = u - [u] - \frac{1}{2} = B_1(u - [u])$, then $f^{(-j)}(u) = (1/(j+1)!)B_{j+1}(u - [u])$ are the successive Bernoulli polynomials, and

$$(1.35) \quad f^{(-j)}(0) = \frac{1}{(j+1)!} B_{j+1} = -\frac{1}{j!} \zeta(-j).$$

(Alternately, we have $f(u) = -\frac{1}{2\pi i} \sum_n \frac{1}{n} e^{2\pi i n u}$, and so by (1.34):

$$f^{(-j)}(0) = -\frac{1}{(2\pi i)^{j+1}} \sum_{0 \neq n \in \mathbf{Z}} \frac{1}{n^{j+1}} = -\frac{1}{j!} \zeta(j)$$

by the functional equation for $\zeta(s)$.)

Differentiating (1.4) and successively integrating by parts, we have

$$\begin{aligned} \psi(x) &= \log x - \frac{1}{2x} + \int_0^\infty \frac{f(u)}{(u+x)^2} du \\ &= \log x - \frac{1}{2x} - \frac{f^{(-1)}(0)}{x^2} + 2 \int_0^\infty \frac{f^{(-1)}(u)}{(u+x)^3} du \\ &= \dots = \log x - \frac{1}{2x} - \frac{f^{(-1)}(0)}{x^2} - \frac{2f^{(-2)}(0)}{x^3} \\ &\quad - \dots - \frac{j!f^{(-j)}(0)}{x^{j+1}} + (j+1)! \int_0^\infty \frac{f^{(-j)}(u)}{(u+x)^{j+2}} du. \end{aligned}$$

Making obvious estimates for the integral and using (1.35), we see that

$$(1.36) \quad \log x + \sum_{j=0}^{\infty} \frac{\zeta(-j)}{(-x)^{j+1}}$$

is an asymptotic series for $\psi(x)$.

II. Let ρ be as before, but now modify f_ρ by setting $f_\rho(u) = B_{1,\rho} + \sum_{a=0}^{[u]} \rho(a)$, where $B_{1,\rho} = (1/d) \sum_{a=0}^{d-1} a\rho(a)$; in that way we have (1.33). As in Example I, we first

rewrite $\int g(u+x)df(u)$ as $\int g'(u+x)f(u)du$ and then continue integrating by parts j more times:

$$\begin{aligned}\psi_\rho(x) &= -\int_0^\infty \frac{df_\rho(u)}{u+x} = \frac{B_{1,\rho}}{x} - \int_0^\infty \frac{f_\rho(u)}{(u+x)^2} du \\ &= \frac{B_{1,\rho}}{x} + \frac{f_\rho^{(-1)}(0)}{x^2} + \cdots + \frac{j!f_\rho^{(-j)}(0)}{x^{j+1}} - (j+1)! \int_0^\infty \frac{f_\rho^{(-j-1)}(u)}{(u+x)^{j+2}} du.\end{aligned}$$

We find, as in Example I, that $\psi_\rho(x)$ has asymptotic series

$$(1.37) \quad \psi_\rho(x) \sim \sum_{j=0}^{\infty} \frac{L(-j, \rho)}{(-x)^{j+1}}.$$

III. Let $f(u) = u - [u] - \frac{1}{2}$ be as in Example I, and let

$$g(u) = \log q \frac{q^u}{1 - q^u},$$

where q is a parameter, $|q| < 1$. By (1.13) we have

$$\begin{aligned}\psi_q(x) &= -\log(1-q) + \int_0^\infty g(u+x)d[u] \\ &= -\log(1-q) - \int_0^\infty g(u+x)df(u) + \int_0^\infty g(u+x)du \\ &= -\log(1-q) + \log(1-q^x) - \int_0^\infty g(u+x)df(u) \\ &= \log \frac{1-q^x}{1-q} + \frac{1}{2}g(x) + \int_0^\infty g'(u+x)f(u)du.\end{aligned}$$

Proceeding just as in Example I, we obtain

$$(1.38) \quad \psi_q(x) \sim \log \frac{1-q^x}{1-q} + \sum_{j=0}^{\infty} \frac{\zeta(-j)}{j!} g^{(j)}(x),$$

where

$$g^{(j)}(x) = (\log q)^{j+1} \left(t \frac{d}{dt} \right)^j \frac{t}{1-t} \Big|_{t=q^x} = (\log q)^{j+1} P_{j+1}(q^x).$$

This is D. Moak's [20] asymptotic series for ψ_q . (Note that our use of the notation P_j is slightly different from Moak's.)

IV. Let $f(u) = f_\rho(u)$ be as in Example II, and let $g(u) = (\log q)(q^u/(1-q^u))$. Following the procedure in Example III, we find the asymptotic series,

$$(1.39) \quad \psi_{q,\rho}(x) \sim \sum_{j=0}^{\infty} \frac{L(-j, \rho)}{j!} g^{(j)}(x).$$

Note that in Examples III and IV, as $q \rightarrow 1^-$ we have

$$\frac{1}{j!} g^{(j)}(x) \rightarrow \frac{1}{j!} \frac{d^j}{dx^j} \left(-\frac{1}{x} \right) = \frac{1}{(-x)^{j+1}},$$

so we obtain the asymptotic series in Examples I and II, as expected.

2. The p -adic case. Let $\mathbf{Q}_p$ denote the field of p -adic numbers, $\mathbf{Z}_p$ denote the ring of p -adic integers, $\mathbf{Z}_p^* = \mathbf{Z}_p - p\mathbf{Z}_p$, $a + (p^N) = \{x \in \mathbf{Z}_p \mid x \equiv a \pmod{p^N}\}$, and $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$. The p -adic absolute value $|\cdot|_p$ on $\mathbf{C}_p$ is normalized so that $|p|_p = 1/p$.

The compact-open subsets of $\mathbf{Z}_p$ are finite disjoint unions of sets of the form $a + (p^N)$. A $\mathbf{C}_p$ -valued measure μ on $\mathbf{Z}_p$ is a bounded, finitely additive map from compact-open subsets to $\mathbf{C}_p$. The integral of a continuous function $g: \mathbf{Z}_p \rightarrow \mathbf{C}_p$ with respect to μ , defined as the usual limit of Riemann sums, always exists. These facts are easy to prove; for details, see, for example, [12, Chapter 2].

Let $z \in \mathbf{C}_p$ be outside the open unit disc about 1: $|z - 1|_p \geq 1$. Define

$$\mu_z(a + (p^N)) = z^a / (1 - z^{p^N}),$$

where a is the least nonnegative integer representative: $0 \leq a < p^N$. Then it is easy to prove [14, Chapter 2] that:

(1) μ_z extends to a measure on $\mathbf{Z}_p$, with $|\mu_z(U)|_p \leq 1$ for all compact-open U .

(2) If z is a root of unity, $\rho(j) = z^j$ (see Examples II and IV in §1), and $L(s, z) = L(s, \rho) = \sum_{n=1}^{\infty} z^n / n^s$ (continued analytically onto the complex s -plane), then the number $L(-k, z) \in \mathbf{Q}(z)$ is given by the p -adic formula

$$(2.1) \quad L(-k, z) = \int_{\mathbf{Z}_p} x^k d\mu_z(x).$$

(3) If $\omega: \mathbf{Z}_p^* \rightarrow 1^{1/(p-1)}$ is the Teichmüller character (considered to be either $\mathbf{C}$ -valued or $\mathbf{C}_p$ -valued; we suppose we have fixed imbeddings of the algebraic numbers in both $\mathbf{C}$ and $\mathbf{C}_p$), $\langle x \rangle = x/\omega(x)$ for $x \in \mathbf{Z}_p^*$, and

$$L^*(s, \omega^k, z) = \sum_{\substack{n \geq 1, p \nmid n}} \omega^k(n) \frac{z^n}{n^s},$$

then the p -adic function

$$(2.2) \quad L_p(s, \omega^j, z) = \int_{\mathbf{Z}_p^*} \langle x \rangle^{-s} \omega^{j-1}(x) d\mu_z(x),$$

$|z - 1|_p \geq 1$, $s \in \mathbf{Z}_p$, when z is a root of unity interpolates the algebraic values

$$L^*(-k, \omega^{j-k-1}, z) = \int_{\mathbf{Z}_p^*} x^k \omega^{j-k-1}(x) d\mu_z(x).$$

(4) If a function $\varphi: \mathbf{C}_p \rightarrow \mathbf{C}_p$ is defined by $\varphi(x) = \int_{\mathbf{Z}_p} g(u+x) d\mu_z(u)$ (where $g: \mathbf{C}_p \rightarrow \mathbf{C}_p$ is continuous; in examples, φ and g might only be defined on a subset of $\mathbf{C}_p$), then φ satisfies

$$(2.3) \quad z\varphi(x+1) - \varphi(x) = -g(x).$$

(5) If $\varphi^*(x)$ is defined by $\varphi^*(x) = \int_{\mathbf{Z}_p^*} g(u+x) d\mu_z(u)$, then

$$(2.4) \quad z^p \varphi^*(x+p) - \varphi^*(x) = - \sum_{a=1}^{p-1} z^a g(x+a).$$

A basic example of the construction of φ and φ^* occurs when $g(u) = -1/u$. For any $z \in \mathbb{C}_p$ with $|z - 1|_p \geq 1$, we define (see [13 or 14])

$$(2.5) \quad \psi_{p,z}(x) = - \int_{\mathbb{Z}_p} \frac{1}{u+x} d\mu_z(u), \quad x \in \mathbb{C}_p - \mathbb{Z}_p;$$

$$(2.6) \quad \psi_{p,z}^*(x) = - \int_{\mathbb{Z}_p^*} \frac{1}{u+x} d\mu_z(u), \quad x \in \mathbb{C}_p - \mathbb{Z}_p^*.$$

By (2.3), $\psi_{p,z}$ satisfies the relationship

$$(2.7) \quad z\psi_{p,z}(x+1) - \psi_{p,z}(x) = 1/x$$

(compare with (1.11)). It is also easy to check that $\psi_{p,z}$ satisfies the “Gauss multiplication formula”

$$(2.8) \quad \psi_{p,z}(x) = \frac{1}{m} \sum_{h=0}^{m-1} z^h \psi_{p,z^m} \left(\frac{x+h}{m} \right),$$

for any positive integer m , and the “Euler parity relation”

$$(2.9) \quad \psi_{p,z}(x) = z^{-1} \psi_{p,z^{-1}}(1-x).$$

The function $\psi_{p,z}^*(x)$ relates to $\psi_{p,z}(x)$ as follows (“removing the p -Euler factor”):

$$\psi_{p,z}^*(x) = \psi_{p,z}(x) - \frac{1}{p} \psi_{p,z^p} \left(\frac{x}{p} \right).$$

REMARK. Compare with Example II of §1 when $\rho(z) = z^j$. In the classical case the measure $df_z(u)$ gives the integer $u = a$ the point mass z^a . In the p -adic case, if we take $|z|_p < 1$, then as $N \rightarrow \infty$, the measure of the interval $a + (p^N)$ around a has measure $\mu_z(a + (p^N)) = z^a/(1 - z^{p^N})$, which approaches z^a ; that is, formally the measure $\mu_z(u)$ also gives $u = a$ point mass z^a . Thus, for $|z|_p < 1$ the p -adic construction is formally the same as the classical one: $\psi_z(x)$ and $\psi_{p,z}(x)$ are given by the same series $-\sum_{a=0}^{\infty} z^a/(x+a)$. However, the functions ψ_z and $\psi_{p,z}$ are of arithmetic interest (relate to Dirichlet L -series or modular forms, see below) only when z is a root of unity. In other words, we must extend analytically beyond the disc where ψ_z and $\psi_{p,z}$ are formally the same in order to reach values of z for which the functions are of number-theoretic interest. Note that $\psi_{p,z}$ and ψ_z satisfy the same relations (2.7) and (2.8).

Now let $q \in \mathbb{C}_p$ be a parameter with $|q - 1|_p < 1$. We want the function q^u to make sense for certain $u \in \mathbb{C}_p$. If $u \in \mathbb{Z}_p$, then q^u converges for any $|q - 1|_p < 1$. More generally, q^u can be defined as $\exp(u \log_p q)$ for any $u \in \mathbb{C}_p$ with $|u|_p < r_q$, where $r_q = |\log_p q|_p^{-1/(p-1)}$. We shall usually assume that $|q - 1|_p < p^{-1/(p-1)}$, in which case

$$r_q = \frac{1}{|q - 1|_p p^{1/(p-1)}} > 1.$$

Let

$$g(u) = \log_p q \frac{q^u}{1 - q^u} = - \frac{d}{du} \log_p \frac{1 - q^u}{1 - q}.$$

Note that $\lim_{q \rightarrow 1} g(u) = -1/u$. Also define

$$g^*(u) = g(u) - g(pu) = \log_p q \left(\frac{q^u}{1 - q^u} - \frac{q^{pu}}{1 - q^{pu}} \right).$$

THEOREM 1. Let $|z - 1|_p \geq 1$, $|q - 1|_p < 1$. The functions

$$\begin{aligned} \psi_{p,q,z}(x) &\stackrel{\text{def}}{=} \int_{\mathbf{Z}_p} g(u+x) d\mu_z(u) \\ &= \log_p q \int_{\mathbf{Z}_p} \frac{q^{u+x}}{1 - q^{u+x}} d\mu_z(u), \quad |x|_p < r_q, x \notin \mathbf{Z}_p; \\ \psi_{p,q,z}^*(x) &\stackrel{\text{def}}{=} \int_{\mathbf{Z}_p^*} g(u+x) d\mu_z(u) \\ &= \log_p q \int_{\mathbf{Z}_p^*} \frac{q^{u+x}}{1 - q^{u+x}} d\mu_z(u), \quad |x|_p < r_q, x \notin \mathbf{Z}_p^*; \\ \psi_{p,q,z}^{**}(x) &\stackrel{\text{def}}{=} \int_{\mathbf{Z}_p^*} g^*(u+x) d\mu_z(u) \\ &= \log_p q \int_{\mathbf{Z}_p^*} \left(\frac{q^{u+x}}{1 - q^{u+x}} - \frac{q^{p(u+x)}}{1 - q^{p(u+x)}} \right) d\mu_z(u), \quad |x|_p < r_q, x \notin \mathbf{Z}_p^*, \end{aligned}$$

satisfy the following relations:

$$(2.10) \quad z\psi_{p,q,z}(x+1) - \psi_{p,q,z}(x) = -\log_p q \frac{q^x}{1 - q^x} = \frac{d}{dx} \log_p \frac{1 - q^x}{1 - q};$$

$$(2.11) \quad \psi_{p,q,z}(x) = \frac{1}{m} \sum_{h=0}^{m-1} z^h \psi_{p,q,z^m} \left(\frac{x+h}{m} \right), \quad \text{for } m = 1, 2, 3, \dots;$$

$$(2.12) \quad \psi_{p,q,z}(x) = z^{-1} \psi_{p,q^{-1},z^{-1}}(1-x);$$

$$(2.13) \quad \psi_{p,q^{-1},z}(x) = \psi_{p,q,z}(x) + \frac{\log_p q}{1 - z};$$

$$(2.14) \quad \psi_{p,q,z}^*(x) = \psi_{p,q,z}(x) - \frac{1}{p} \psi_{p,q^p,z^p}(x/p);$$

$$(2.15) \quad z^p \psi_{p,q,z}^*(x+p) - \psi_{p,q,z}^*(x) = - \sum_{a=1}^{p-1} z^a \log_p q \frac{q^{x+a}}{1 - q^{x+a}};$$

$$(2.16) \quad \psi_{p,q,z}^{**}(x) = \psi_{p,q,z}^*(x) - \frac{1}{p} \psi_{p,q^p,z^p}^*(x).$$

PROOF. (2.10) is an immediate consequence of (2.3). To prove the "Gauss multiplication formula" (2.11), we write the right side as

$$\begin{aligned} &\lim_{N \rightarrow \infty} \frac{1}{m} \sum_{h=0}^{m-1} z^h \log_p q \sum_{j=0}^{p^N-1} \frac{q^{m(j+(x+h)/m)}}{1 - q^{m(j+(x+h)/m)}} \frac{z^{mj}}{1 - z^{mp^N}} \\ &= \lim_{N \rightarrow \infty} \log_p q \sum_{h,j} \frac{q^{mj+h+x}}{1 - q^{mj+h+x}} \frac{z^{mj+h}}{1 - z^{mp^N}}. \end{aligned}$$

If we write $mj + h = p^N k + l$, $k = 0, 1, \dots, m-1$, $l = 0, 1, \dots, p^N - 1$; and if we use the fact that $q^{p^N k + l} \rightarrow q^l$ as $N \rightarrow \infty$, we find that the last limit is equal to

$$\begin{aligned} \lim_{N \rightarrow \infty} \log_p q \sum_{l=0}^{p^N-1} \frac{q^{l+x}}{1-q^{l+x}} z^l \sum_{k=0}^{m-1} \frac{z^{p^N k}}{1-z^{mp^N}} &= \lim_{N \rightarrow \infty} \log_p q \sum_{l=0}^{p^N-1} \frac{q^{l+x}}{1-q^{l+x}} \frac{z^l}{1-z^{p^N}} \\ &= \log_p q \int_{\mathbb{Z}_p} \frac{q^{u+x}}{1-q^{u+x}} d\mu_z(u), \end{aligned}$$

as desired. The proofs of (2.12)–(2.16) are equally straightforward. Q.E.D.

REMARK. The “multiplication formula” (2.11) and the parity relation (2.12) can be combined into the following more general identity:

$$\psi_{p,q,z}(x) = \frac{1}{|m|} \sum z^h \psi_{p,q^m,z^m} \left(\frac{x+h}{m} \right),$$

where the summation is over $\min(m, 0) \leq h < \max(m, 0)$ and m is any positive or negative integer.

Returning to the general situation, $\varphi(x) = \int_{\mathbb{Z}_p} g(u+x) d\mu(u)$, we note that the following two expansions are possible:

$$\begin{aligned} (2.17) \quad \text{for } x \text{ small, } g(u+x) &= \sum_{j=0}^{\infty} \frac{x^j}{j!} g^{(j)}(u) \\ \Rightarrow \varphi(x) &= \sum_{j=0}^{\infty} \frac{x^j}{j!} \int_{\mathbb{Z}_p} g^{(j)}(u) d\mu(u); \end{aligned}$$

$$\begin{aligned} (2.18) \quad \text{for } x \text{ large, } g(u+x) &= \sum_{j=0}^{\infty} \frac{u^j}{j!} g^{(j)}(x) \\ \Rightarrow \varphi(x) &= \sum_{j=0}^{\infty} \frac{g^{(j)}(x)}{j!} \int_{\mathbb{Z}_p} u^j d\mu(u). \end{aligned}$$

The second of these expansions holds whenever the infinite sum converges; since the integral is bounded, we have convergence if $|g^{(j)}(x)/j!|_p \rightarrow 0$. The first expansion holds for $|x|_p < p^{-1/(p-1)}$ provided that the $g^{(j)}(u)$ are defined, continuous, and bounded (uniformly in j) on the support of μ . We can apply the second expansion for $\mu = \mu_z$ and $\mu = \mu_z|_{\mathbb{Z}_p^*}$ and for $g(u) = -1/u$ and $g(u) = (\log_p q)(q^u/(1-q^u))$. We can apply the first expansion for $\mu = \mu_z|_{\mathbb{Z}_p^*}$ and for $g(u) = -1/u$, $g(u) = (\log_p q)(q^u/(1-q^u))$ and

$$g^*(u) = \log_p q \left(\frac{q^u}{1-q^u} - \frac{q^{pu}}{1-q^{pu}} \right).$$

THEOREM 2. Let $\psi_{p,z}(x)$, $\psi_{p,z}^*(x)$, $\psi_{p,q,z}(x)$, $\psi_{p,q,z}^*(x)$ and $\psi_{p,q,z}^{**}(x)$ be the functions defined in (2.5)–(2.6) and in Theorem 1. In (2) below we suppose that $|q-1|_p < p^{-1/(p-1)}$, and in (4) we suppose that $|q-1|_p < p^{-2/(p-1)}$ (i.e., $r_q > p^{1/(p-1)}$). Then

(1) for $|x|_p < 1$,

$$\psi_{p,z}^*(x) = \sum_{j=1}^{\infty} (-1)^j x^{j-1} L_p(j, \omega^{1-j}, z)$$

(compare with (1.12));

(2) for $|x|_p < p^{-1/(p-1)}$,

$$\psi_{p,q,z}^*(x) = \sum_{j=1}^{\infty} x^{j-1} \frac{(\log_p q)^j}{(j-1)!} \int_{\mathbf{Z}_p^*} P_j(q^u) d\mu_z(u),$$

$$\psi_{p,q,z}^{**}(x) = \sum_{j=1}^{\infty} x^{j-1} \frac{(\log_p q)^j}{(j-1)!} \int_{\mathbf{Z}_p^*} P_j^*(q^u) d\mu_z(u),$$

where P_j was defined in (1.15) and

$$P_j^*(q) \stackrel{\text{def}}{=} P_j(q) - p^{j-1} P_j(q^p) = \left(q \frac{d}{dq} \right)^{j-1} \left(\frac{q}{1-q} - \frac{q^p}{1-q^p} \right)$$

(compare with (1.31));

(3) for $|x|_p > 1$,

$$\psi_{p,z}(x) = \sum_{j=0}^{\infty} \frac{L(-j, z)}{(-x)^{j+1}}, \quad \psi_{p,z}^*(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{(-x)^{j+1}}$$

(compare with (1.37)). Here in the first sum $L(-j, z) \stackrel{\text{def}}{=} \int_{\mathbf{Z}_p} u^j d\mu_z(u)$, which coincides with the classical value $L(-j, z)$ when z is a root of unity;

(4) for $p^{1/(p-1)} < |x|_p < r_q$,

$$\psi_{p,q,z}(x) = \sum_{j=0}^{\infty} \frac{L(-j, z)}{j!} (\log_p q)^{j+1} P_{j+1}(q^x),$$

$$\psi_{p,q,z}^*(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{j!} (\log_p q)^{j+1} P_{j+1}(q^x),$$

$$\psi_{p,q,z}^{**}(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{j!} (\log_p q)^{j+1} P_{j+1}^*(q^x)$$

(compare with (1.39)).

The proof of Theorem 2 follows immediately from (2.17), (2.18), (2.1), (2.2) and the following observations concerning convergence: in (1), (3) and (4), $L_p(j, \omega^{1-j}, z)$, $L(-j, z)$, $L_p(-j, \omega^{j+1}, z)$ are bounded with respect to j ; in (2), $(\log_p q)^j P_j(q^u)$ and $(\log_p q)^j P_j^*(q^u)$ are bounded over $u \in \mathbf{Z}_p^*$ uniformly with respect to j ; and in (4), we have

$$|(\log_p q)^{j+1} P_{j+1}(q^x)|_p \leq \left| \frac{\log_p q}{1-q^x} \right|_p^{j+1} = \left| \frac{1}{x} \right|_p^{j+1},$$

and the same relation for $|(\log_p q)^{j+1} P_{j+1}^*(q^x)|_p$.

REMARK. The integrals $a_j = a_{j,q,z}$ in the coefficients of the Taylor series for $\psi_{p,q,z}^{**}(x)$ (part (2) of Theorem 2) are related to p -adic Eisenstein series, but more indirectly than in the classical case. To see this connection, first suppose that

$|z|_p < 1$. In that case

$$a_j = a_{j,q,z} = \int_{\mathbf{Z}_p^*} P_j^*(q^u) d\mu_z(u) = \sum_{m \geq 1, p \nmid m} P_j^*(q^m) z^m.$$

If we also suppose that $|q|_p < 1$, then we have

$$P_j^*(q) = \sum_{n \geq 1, p \nmid n} n^{j-1} q^n$$

and

$$\sum_{m \geq 1, p \nmid m} P_j^*(q^m) z^m = \sum_{m \geq 1, p \nmid m} q^m \sum_{dd'=m} d^{j-1} z^{d'}.$$

Now set z equal to a nontrivial D th root of unity, and define f on $\mathbf{Z}/D\mathbf{Z} \times \mathbf{Z}/D\mathbf{Z}$ by $f(u, v) = z^v$ (f does not depend on u). Then

$$a_{j,q,z} + (-1)^j a_{j,q,z^{-1}} = \sum_{m \geq 1, p \nmid m} q^m \sum_{dd'=m} d^{j-1} (f(d, d') + (-1)^j f(-d, -d'))$$

is the series Katz denotes $2\Phi_{j-1,0,f}^*$ (see (6.4.1) of [8]; note that the definitions and results generalize trivially to $\mathbf{Z}_p[z]$ -valued functions f). It is a p -adic modular form for the congruence-subgroup $\Gamma(D)$ (see [8]).

In other words, we can consider the power series

$$a_j = \sum_{m \geq 1, p \nmid m} q^m \sum_{dd'=m} d^{j-1} z^{d'} \in \mathbf{Z}[z][[q]] \subset \mathbf{Z}[[z, q]]$$

as a p -adic analytic function in the two variables z, q for $|z|_p \leq 1, |q|_p < 1$. When $z^D = 1$ we essentially have a p -adic Eisenstein series. On the other hand, for any fixed z with $|z|_p < 1$, the same series a_j has a unique analytic continuation

$$a_j = \sum_{m \geq 1, p \nmid m} P_j^*(q^m) z^m$$

to the region $\{q \in \mathbf{C}_p \mid |q|_p \leq 1, q^n \neq 1 \text{ for all } n, p^2 \nmid n\}$. This p -adic analytic function is given on the region $0 < |q-1|_p < p^{-1/(p-1)}$ by the integral $\int_{\mathbf{Z}_p^*} P_j^*(q^u) d\mu_z(u)$. Finally, for any fixed q with $0 < |q-1|_p < p^{-1/(p-1)}$, this integral gives a unique analytic continuation (in z) to the region $|z-1|_p \geq 1$. In particular, for z a D th root of unity we obtain essentially the j th Taylor expansion coefficient of the twisted psi-function $\psi_{p,q,z}^{**}$. This is a p -adic analogy to the occurrence discussed in §1 of classical Eisenstein series in the expansion of $\psi_q(x+1)$ and $\psi_{q,z}(x+1)$.

Finally, we discuss the relationship between $\psi_{p,q,z}$ and the q -extensions defined in [15] for J. Diamond's p -adic log-gamma function [5]. We temporarily use $\tilde{\psi}$ to denote the functions defined in [15] by means of limits of the form

$$\lim_{N \rightarrow \infty} \frac{1}{dp^N} \sum_{0 \leq j < dp^N} g(x+j).$$

In what follows Σ' means that indices divisible by p are omitted. Thus for $z^d = 1$ (with $z \neq 1$, $p \nmid d$) and for $|x|_p < r_q$, $x \notin \mathbf{Z}_p$,

$$\begin{aligned}\tilde{\psi}_{p,q,z}(x) &= \lim_{\text{def } N \rightarrow \infty} \frac{1}{dp^N} \sum_{0 \leq j < dp^N} z^j \log_p \frac{1 - q^{x+j}}{1 - q}; \\ \tilde{\psi}_{p,q,z}^*(x) &= \lim_{\text{def } N \rightarrow \infty} \frac{1}{dp^N} \sum'_{0 \leq j < dp^N} z^j \log_p \frac{1 - q^{x+j}}{1 - q} \\ &= \tilde{\psi}_{p,q,z}(x) - \frac{1}{p} \tilde{\psi}_{p,q^p,z^p}(x/p).\end{aligned}$$

THEOREM 3. (1) $\tilde{\psi}_{p,q,z}(x) = \psi_{p,q,z}(x)$ for $|x|_p < r_q$, $x \notin \mathbf{Z}_p$.

(2) $\tilde{\psi}_{p,q,z}^*(x) = \psi_{p,q,z}^*(x)$ for $|x|_p < r_q$, $x \notin \mathbf{Z}_p^*$.

The proof is similar to the proof that $\tilde{G}_{p,z}(x) = G_{p,z}(x)$ in [14, p. 51]. Namely, one first shows that (2.11) also holds for $\tilde{\psi}$. Then we can use (2.11) with $m = p^n$ for both ψ and $\tilde{\psi}$ to see that it suffices to prove that

$$\tilde{\psi}_{p,q^{p^n},z^{p^n}}\left(\frac{x+h}{p^n}\right) = \psi_{p,q^{p^n},z^{p^n}}\left(\frac{x+h}{p^n}\right),$$

i.e., to prove that $\tilde{\psi}_{p,q,z}(x) = \psi_{p,q,z}(x)$ for $|x|_p$ large. But this is true because both sides have the same Stirling series. The proof that $\psi^* = \psi^*$ is similar.

REFERENCES

1. G. E. Andrews, *The theory of partitions*, Encyclopedia of Math. and its Applications, vol. 2, Addison-Wesley, Reading, Mass., 1976.
2. E. Artin, *The gamma function*, Holt, Rinehart and Winston, New York, 1964.
3. R. Askey, *The q -gamma and q -beta functions*, Applicable Anal. **8** (1978), 125–141.
4. ———, *Ramanujan's extensions of the gamma and beta functions*, Amer. Math. Monthly **87** (1980), 346–359.
5. J. Diamond, *The p -adic log gamma function and p -adic Euler constants*, Trans. Amer. Math. Soc. **233** (1977), 321–337.
- 5a. ———, *On the values of p -adic L -functions at positive integers*, Acta Arith. **35** (1979), 223–237.
6. K. Iwasawa, *Lectures on p -adic L -functions*, Princeton Univ. Press, Princeton, N. J. 1972.
7. N. Katz, *p -adic properties of modular schemes and modular forms*, Modular Functions of One Variable III, Lecture Notes in Math., vol. 350, Springer-Verlag, New York, 1973, pp. 69–190.
8. ———, *p -adic interpolation of real analytic Eisenstein series*, Ann. of Math. **104** (1976), 459–571.
9. ———, *The Eisenstein measure and p -adic interpolation*, Amer. J. Math. **99** (1977), 238–311.
10. ———, *p -adic L -functions for CM fields*, Invent. Math. **49** (1978), 199–297.
11. ———, *p -adic L -functions, Serre-Tate local moduli, and ratios of solutions of differential equations*, Proc. Internat. Congr. Math. (Helsinki, 1978), Accad. Sci. Fenn., Helsinki, 1980.
12. N. Koblitz, *p -adic numbers, p -adic analysis, and zeta-functions*, Springer-Verlag, New York, 1977.
13. ———, *A new proof of certain formulas for p -adic L -functions*, Duke Math. J. **46** (1979), 455–468.
14. ———, *p -adic analysis: A short course on recent work*, London Math. Soc. Lecture Note Series 46, Cambridge Univ. Press, Cambridge, 1980.
15. ———, *q -extensions of the p -adic gamma function*, Trans. Amer. Math. Soc. **260** (1980), 449–457.
16. M. Krasner, *Rapport sur le prologement analytique dans les corps valués complets par la méthode des éléments analytiques quasi-connexes*, Bull. Soc. Math. France Mém. **39–40** (1974), 131–254.
17. T. Kubota and H. W. Leopoldt, *Eine p -adische Theorie der Zetawerte*. I, J. Reine Angew. Math. **214–215** (1964), 328–339.
18. S. Lang, *Introduction to modular forms*, Springer-Verlag, New York, 1976.
19. ———, *Cyclotomic fields*, Springer-Verlag, New York, 1978.

20. D. Moak, University of Wisconsin Dissertation, Madison, Wis., 1980.
21. J.-P. Serre, *Formes modulaires et fonctions zêta p -adiques*, Modular Functions of One Variable. III, Lecture Notes in Math., vol.350, Springer-Verlag, New York, 1973, pp. 191–268.
22. ———, *A course in arithmetic*, Springer-Verlag, New York, 1973.
23. G. N. Watson and E. T. Whittaker, *A course of modern analysis*, 4th ed., Cambridge Univ. Press, Cambridge, 1927.

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195 (Current address)